vious definition of pervasiveness in the general case. Then corollary 3 of Pringle's note is:

Theorem C: If the Hamiltonian H is independent of time and if the power dissipation function P is pervasive, then the dynamical system is asmyptotically stable if H is positive definite in a region S around the origin in the (q_i, p_i) space and unstable if H is indefinite or negative definite in S.

Theorem C gives an almost immediate proof of the KTC theorem, Theorem A, which is simpler than the proofs in Refs. 1 and 2. If $\mathbf{D}$ is positive definite, $P \equiv 0$ implies $\dot{\mathbf{x}} \equiv 0$. But then by (2), $\mathbf{Kx} = 0$, and hence $\mathbf{x} = 0$, since we have assumed that $\mathbf{K}$ has no zero eigenvalues. Therefore P is pervasive. Since $\mathbf{A}$ is assumed positive definite, the Hamiltonian $H = \frac{1}{2}[\dot{\mathbf{x}}, \mathbf{A}\dot{\mathbf{x}} + \mathbf{x}, \mathbf{Kx}]$ is positive definite with respect to $q_i = x_i$, $p_i = \dot{x}_i$ if $\mathbf{K}$ has all positive eigenvalues and indefinite if $\mathbf{K}$ has some negative eigenvalues, which proves Theorem A.

Moreover, Theorem C is not restricted to linear systems. For example, for $d_0 = 0$, $g \neq 0$, damping terms in (1) of the form $d_1(\dot{x}_1 - \dot{x}_2)$ or $d_1|\dot{x}_1 - \dot{x}_2|(\dot{x}_1 - \dot{x}_2)$ both give a pervasive power function P. In either case, the stability of system (1) would be completely determined by the signs of the k's.

Likewise, Theorem C suggests that Theorem B will remain valid if the assumption of pervasiveness is used. This is indeed the case; we have:

Theorem D: If in system (2) $P = -(\dot{\mathbf{x}}, \mathbf{D}\dot{\mathbf{x}})$ is pervasive, Eq. (3) has no roots on the imaginary axis and as many roots in the closed right half-plane as there are negative eigenvalues of $\mathbf{K}$.

The proof is very similar to the proof of Theorem II of Ref. 2 and will only be sketched here. Let $\mathbf{x} = \mathbf{V} \exp st$, where $\mathbf{V}$ is an eigenvector satisfying the matrix equation $(s^2\mathbf{A} + s\mathbf{D} + s\mathbf{G} + \mathbf{K})\mathbf{V} = 0$. First, one shows that in a pervasive system, $(\mathbf{V}^*, \mathbf{DV}) = 0$ implies that $\mathbf{V} = 0$ [$\mathbf{V}^*$ is the complex conjugate of $\mathbf{V}$ and $(\mathbf{V}^*, \mathbf{DV}) = \Sigma V_i^* V_j D_{ij}$]. To do this, let $\mathbf{V} = \mathbf{Q}\boldsymbol{\xi}$, where $\mathbf{Q}$ is a matrix[4] diagonalizing $\mathbf{D}$ into the form: $(d_{11}, d_{22}, \ldots, d_{mm}, 0, 0, \ldots)$, and let $\mathbf{x} = \mathbf{V} \exp st + \mathbf{V}^* \exp s^*t$. Then we have the following chain of arguments, where $\rightarrow$ denotes "implies":
$(\mathbf{V}^*, \mathbf{DV}) = 0 \rightarrow \xi_1 = \xi_2 = \ldots = \xi_m = 0 \rightarrow (\mathbf{V}, \mathbf{DV}) = (\mathbf{V}^*, \mathbf{DV}^*) = 0 \rightarrow \mathbf{x}, \mathbf{Dx} = 0 \rightarrow \mathbf{x} = 0 \rightarrow \mathbf{V} = 0$. Hence, for any nonzero eigenvector, $(\mathbf{V}^*, \mathbf{DV}) > 0$.

Next we define $\hat{\mathbf{D}}$ and $\hat{\mathbf{G}}$ by $\hat{\mathbf{G}} = r\mathbf{G}$ and $\hat{\mathbf{D}} = r\mathbf{D} + (1 - r)\mathfrak{D}$, where $0 \leq r \leq 1$, and $\mathfrak{D}$ is a diagonal matrix consisting of ones where the corresponding diagonal elements of $\mathbf{D}$ are zero and of the diagonal elements themselves where they are nonzero. As in Ref. 2, when r goes from 0 to unity, the roots of the characteristic equation do not cross the imaginary axis, and hence there are as many roots in the right half-plane as there are negative eigenvalues of $\mathbf{K}$.

Finally, we note that in system (2) we may have a positive semidefinite $\mathbf{D}$ but a nonpervasive power function. For such systems Theorems A–D do not apply. However, the following result may apply (see Theorem III of Ref. 2):

Theorem E: If $\mathbf{D}$ *is positive semidefinite in system (2) Eq. (3) will have all its roots in the closed left half-plane if all the eigenvalues of* $\mathbf{K}$ *are positive, and at least one root in the open right half-plane if all the eigenvalues of* $\mathbf{K}$ *are negative.*

Thus by Theorem *E* if $d_0 = g = 0$ in (1) and $k_1/m_1 = k_2/m_2$, all of the four roots are on the imaginary axis or in the left half-plane if $k_1 > 0$, and there is at least one root in the open right half-plane if $k_1 < 0$.

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Chemical Nonequilibrium Studies in Supersonic Nozzle Flows

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Introduction

In many cases an accurate analysis of nozzle flow fields requires the inclusion of two-dimensional effects and finite rate chemistry. The theoretical model for incorporating chemical reactions in two-dimensional inviscid supersonic nozzle flows is available[1] and numerical solutions have been obtained.[2–4] In Ref. 2 an approximation is presented wherein the nonequilibrium terms are neglected along Mach lines but retained along streamlines. The justification for this assumption was that the Mach line characteristics are used only to determine the pressure and geometry, and the pressure does not vary significantly between the two extremes of equilibrium and frozen flow. The approximation, however, leads to a small reduction in pressure, which, in many cases, is of the same order of magnitude as the reduction in pressure due to nonequilibrium effects. Hence there exists the possibility of obscuring the nonequilibrium effects in one of the most important flow field parameters for thrust and drag calculations, namely the wall pressure distribution. The numerical procedure adopted in Ref. 4 is unrelated to the characteristics, and hence the authors indicate caution in dealing with flows where wave phenomena are an important feature. Waves are always an important feature in nozzle flows. For example, even simple conical nozzles can lead to shock formation,[5,6] and many propulsion and wind-tunnel nozzles are characterized by a rapid initial expansion section.[7] Some form of precise differencing technique is required in all of the preceding methods when the flow parameters are near their equilibrium values. Thus complicated and lengthy numerical integration techniques are required to achieve an acceptable degree of accuracy. A review of some of the past experimental and analytical studies with one-dimensional flow fields[8–11] suggests that a two-dimensional analysis should be

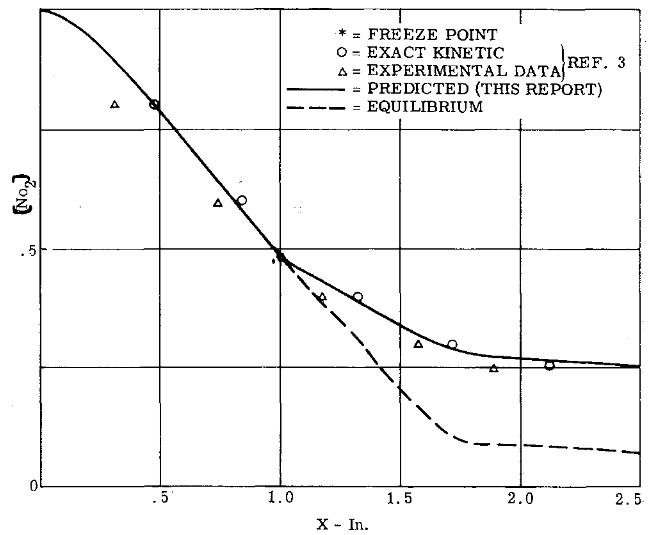

Fig. 1 Comparison with experimental data.

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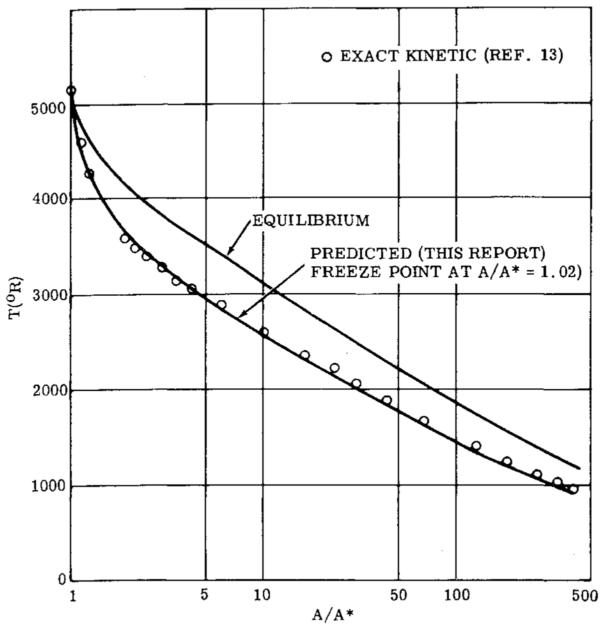

Fig. 2 Comparison with exact analysis.

amenable to considerable simplification and yet accurately predict the significant flow variables.

Method

The method adopted here utilizes a standard (equilibrium or frozen) method of characteristics combined with Bray's analysis[8] for determining freeze points along streamlines. For a multicomponent mixture the modification proposed in Ref. 9 is used. Streamlines are computed at each point in the characteristic grid, and a sudden freeze point criterion is applied in the following form:

$$\frac{d}{dt}\left(\frac{1}{Wm}\right) \overset{f}{\underset{e}{\gtrless}} \sum_{j} R_r^{\,j}$$

where Wm is the equilibrium molecular weight, and R_r^j is the recombination rate to be summed over all j reactions that contribute to the net production of moles. Frozen flow f is indicated by the upper inequality and equilibrium flow e by lower inequality. By allowing different freeze point locations, consistent with the local rate of expansion, the two-dimensionality of the flow field is preserved. Further details on the numerical construction, rate equations, and rate constants employed are contained in Ref. 12.

Table 1 Integrated pressure defects[a]

Propellant	Model	Percent loss in supersonic thrust
$H_2 + F_2$ $O/F = 10$	A	25.6
	B	8.4
$H_2 + O_2$ $O/F = 5$	A	8.6
	B	4.5
$N_2H_4 + F_2$ $O/F = 1.5$	A	20.4
	B	12.7

X, in.	Nozzle contour Radius, in.
0.	1.00
0.11	1.033
0.25	1.11
0.50	1.26
1.00	1.56
2.00	2.16
4.00	3.18
8.00	4.70
12.00	5.81
14.37	6.32

[a] Chamber pressure = 300 psia; O/F = oxidizer/fuel weight ratio.

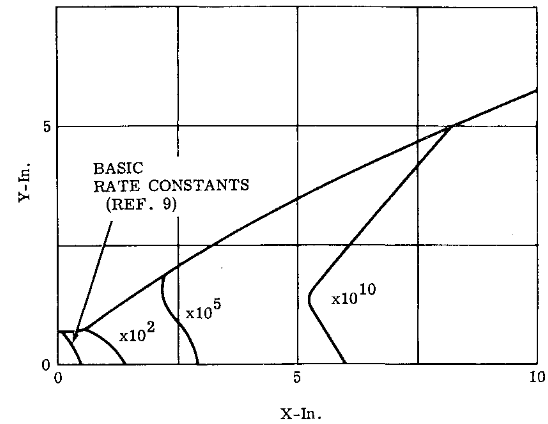

Fig. 3 Predicted freeze lines.

Results

The results of the approximation proposed here compare well with experimentally determined concentration profiles[3] in a planar conical nozzle, as shown in Fig. 1. In this figure, $[NO_2]$ represents the concentration normalized with respect to the throat value and X the distance from the throat. The data shown in Fig. 1 are for the streamline along the plane of symmetry (the chamber conditions are 33 psia, 770°R, 0.97 mole fraction N_2, and 0.03 mole fraction NO_2).‡ Additional comparisons for the wall streamline[12] also show good agreement. Excellent agreement is obtained when compared to the results of an exact analysis[13] in a high expansion area ratio A/A^* nozzle (Fig. 2). The inviscid wall temperature T is used as a basis for comparison in Fig. 2; the propellants are hydrogen and oxygen at 60 psia chamber pressure and an equivalence ratio of 1.59. The shape of the freeze lines for this nozzle (Fig. 3) was obtained by varying the basic rate constants.[9] It is interesting to note the reversal in the slope of the freeze line as freezing is delayed (increased rate constants). This reversal is indicative of the shift in region of maximum temperature gradients from the contour to the axis of symmetry as the flow proceeds downstream of the throat.

An indication of the maximum error involved with the approximation proposed in Ref. 2 was obtained in the following manner. The frozen sound speed was used along Mach line characteristics, but the flow properties were kept at their equilibrium values along streamlines (model A in Table 1). Thus only the effects of the approximation are obtained. As a basis for comparison, the integrated pressure distribution downstream of the throat in a 40:1 expansion area ratio nozzle was obtained with the kinetic approximation proposed here (model B). Both pressure distributions were referenced to the integrated pressure distribution for completely equilibrium flow, and the percent deviation is shown in Table 1. It will be noted that the approximation of Ref. 2 leads to pressure defects larger than the defects due to the nonequilibrium nature of the flow.

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‡ The input data are described in Table 6 of Ref. 12 and are also available in Refs. 3 and 13.

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An MHD Channel Flow with Temperature Dependent Electrical Conductivity

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Magnetohydrodynamic channel flows have received considerable attention of late, and several solutions using approximate methods have been obtained.[1–4] The motivation given for some of these solutions has been the current interest in MHD power generators. However the models used have been appropriate only for certain liquid metal flows since the assumptions made included 1) an incompressible fluid with constant properties and 2) zero Hall parameter. These assumptions will not be valid in a useful gaseous MHD generator, and it is necessary to consider the effect of relaxing these restrictions before extrapolating these results to channel flows of an electrically conducting gas. The effect of the Hall parameter has been considered by Tani,[5] and here we will consider the effect of a temperature-dependent electrical conductivity.

Hale and Kerrebrock[6] have considered a compressible boundary layer with temperature-dependent fluid properties in an MHD accelerator and have shown that both velocity and temperature overshoots (velocities and temperatures greater than the freestream values) exist in the boundary layer on the insulating wall. It is the purpose of this work to demonstrate by analyzing a particularly simple model how these peculiar velocity distributions can occur even when the temperature distribution is approximately parabolic, and the effect of the MHD terms in the energy equation is small.

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This is in contrast to the results of Hale and Kerrebrock and an earlier note by Lykoudis[7] where the velocity overshoots are a direct result of the temperature (or conductivity) overshoots inside the boundary layer. A channel flow is chosen as the model since at high Hartmann numbers this illustrates several features of a boundary layer in that viscous forces are only important close to the channel walls.

Problem Formulation

Consider the steady fully-developed flow of an electrically conducting gas through a channel of rectangular cross section as shown in Fig. 1. It is assumed that $a \gg 2h$ so that the flow is two-dimensional; that the specific heat c_p, viscosity μ, thermal conductivity k, Prandtl number Pr, and density ρ of the gas are constant; that E and B, the imposed electric and magnetic fields are constant; and that the magnetic Reynolds number is small. For these assumptions the x momentum equation reduces to

$$\mu(d^2u/dy^2) - dp/dx - jB = 0 \tag{1}$$

where the current density j which flows in the z direction is given by

$$j = \sigma(uB + E) \tag{2}$$

where σ is the electrical conductivity, which is a function of temperature. The energy equation reduces to

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{j^2}{\sigma} + u\frac{dp}{dx} + k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{du}{dy}\right)^2 \tag{3}$$

The dimensionless temperature $\theta = (T - T_w)/(T_c - T_w)$ is now introduced and is assumed to be a function of y only; subscripts w and c refer to wall and centerline values, respectively. The electrical conductivity is assumed to be given by the relation

$$\sigma/\sigma_c = \theta^\omega \tag{4}$$

where ω is a positive exponent. Equations (1) and (3) can now be expressed in the following dimensionless form

$$\frac{1}{M^2}\frac{d^2u'}{d\eta^2} - \frac{dP}{d\xi} - \theta^\omega(u' - K) = 0 \tag{5a}$$

$$\frac{1}{M^2 mPr}\frac{d^2\theta}{d\eta^2} + \frac{1}{M^2}\left(\frac{du'}{d\eta}\right)^2 + \theta^\omega(u' - K)^2 - u'\left(\frac{dT'}{d\xi} - \frac{dP}{d\xi}\right) = 0 \tag{5b}$$

where η and ξ are the dimensionless coordinates y/h and $x\sigma_c B^2/(\rho u_c)$, respectively; $u' = u/u_c$ is the dimensionless velocity, $M = Bh(\sigma_c/\mu)^{1/2}$ is the Hartmann number; $P = p/(\rho u_c^2)$ is the dimensionless pressure; $K = -E/(u_c B)$ is the loading factor; Pr is the Prandtl number; $m = u_c^2/[c_p(T_c - T_w)]$; and $T' = c_p T_w/u_c^2$ is the dimensionless wall temperature. It will be assumed that σ_c is constant. This implies that $dT_c/dx \ll (T_c - T_w)/h$.

The boundary conditions for Eqs. (5) are that at the wall $u_w' = 0$ and $\theta_w = 0$; at the centerline $du'/d\eta = 0$ and $d\theta/$

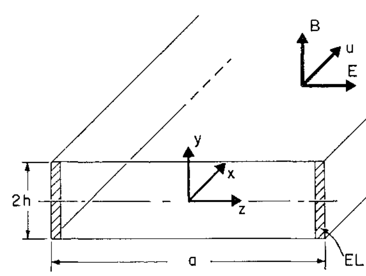

Fig. 1. Schematic of channel showing coordinate system.